Dept. of Math. University of Oslo
Pure Mathematics No. 22
ISSN 0806–2439 June 2004

Forward integrals and an Itô formula for fractional Brownian motion

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Revised in April 2007*

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Abstract

We consider the *forward integral* with respect to fractional Brownian motion $B^{(H)}(t)$ and relate this to the Wick-Itô-Skorohod integral by using the M-operator introduced by [10] and the Malliavin derivative $D_t^{(H)}$. Using this connection we obtain a general Itô formula for the Wick-Itô-Skorohod integrals with respect to $B^{(H)}(t)$, valid for $H \in (\frac{1}{2}, 1)$.

AMS 2000 subject classifications: Primary 60G15, 60G18, 60HXX, 60H07, 60H40.

Key words and phrases: Forward integral, Wick-Itô-Skorohod integral, Wick product, Malliavin derivative, fractional Brownian motion, Itô formula.

*Preprint of an article submitted for consideration in Infinite Dimensional Analysis, Quantum Probability and Related Topics 2008 [copyright World Scientific Publishing Company] [http://www.worldscinet.com/idaqp/idaqp.shtml]

1 Introduction

Fractional Brownian motion $B^{(H)}(t) = B^{(H)}(t, \omega)$, $t \geq 0, \omega \in \Omega$, with Hurst parameter $H \in (0, 1)$ is a real-valued Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the property that

$$E\left[B^{(H)}(t)\right] = B^{(H)}(0) = 0 \quad \text{for all} \quad t \geq 0$$

and

$$E\left[B^{(H)}(t)B^{(H)}(s)\right] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]; \quad t, s \geq 0$$

where E denotes expectation with respect to $\mathbb{P}$.
Because of its properties the fractional Brownian motion has been used to model a number of phenomena, e.g. in biology, meteorology, physics and finance. See e.g. [24], [6], [7], [21] and the references therein. In that connection, it is of interest to develop a stochastic calculus based on $B^{(H)}(t)$. In particular, one wants an integration theory, a white noise theory and a Malliavin calculus for such processes. See e.g. [6] and the references therein for an account of this.
There are several different integral concepts of independent interest, among which the *pathwise integral* and the *Wick-Itô-Skorohod integral*. For each of these integrals several versions of an Itô formula have been obtained. See for example [5], [7], [9], [15], [18], [19], [11].
The purpose of this paper is to prove a new general Itô formula for the Wick-Itô-Skorohod integral based on the M-operator of [10] and the Malliavin derivative $D_t^{(H)}$, valid for $H \in (\frac{1}{2}, 1)$.

2 Some preliminaries

Here we recall the approach of [10], [16],[7] to white-noise calculus for fractional Brownian motion.
We begin by recalling the standard setup for the classical white noise probability space. See e.g. [13], [17], [14] or [1] for more details.

Definition 2.1 *Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and let $\Omega := \mathcal{S}'(\mathbb{R})$ be its dual, usually called the space of tempered distributions. Let $\mathbb{P}$ be the probability measure on the Borel sets $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ defined by the property that*

$$\int_{\mathcal{S}'(\mathbb{R})} exp(i < \omega, f >) d\mathbb{P}(\omega) = exp(-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}); \quad f \in \mathcal{S}(\mathbb{R}), \tag{2.1}$$

where $i = \sqrt{-1}$ and $< \omega, f >= \omega(f)$ is the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$.
The measure $\mathbb{P}$ is called the white noise probability measure. Its existence follows from the Bochner–Minlos theorem.

In the following we let

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \ldots \tag{2.2}$$

denote the *Hermite polynomials* and we let

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}; \quad n = 1, 2, \ldots \tag{2.3}$$

be the *Hermite functions*. Then $\xi_n \in \mathcal{S}(\mathbb{R})$. From [25], $\{\xi_n\}_{n=1}^{\infty}$ constitutes an orthonormal basis for $L^2(\mathbb{R})$. Let $\mathcal{J}$ be the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of finite *length* $l(\alpha) = \max\{i; \alpha_i \neq 0\}$, with $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ for all i. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J}$ we put $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and we define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(< \omega, \xi_1 >) h_{\alpha_2}(< \omega, \xi_2 >) \cdots h_{\alpha_n}(< \omega, \xi_n >). \tag{2.4}$$

In particular special cases are the unit vectors

$$\epsilon^{(k)} = (0, 0, \ldots, 0, 1) \tag{2.5}$$

with 1 on the k'th entry, 0 otherwise; $k = 1, 2, \ldots$. We now use the well-known Wiener-Itô chaos expansion Theorem to define the following space $(\mathcal{S})$ of stochastic test functions and the dual space $(\mathcal{S})^*$ of stochastic distributions:

Definition 2.2 a) *We define the Hida space $(\mathcal{S})$ of stochastic test functions to be all $\psi \in L^2(\mathbb{P})$ whose expansion*

$$\psi(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega)$$

satisfies

$$\|\psi\|_k^2 := \sum_{\alpha \in \mathcal{J}} a_\alpha^2 \alpha! (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all } k = 1, 2, \ldots \tag{2.6}$$

where

$$(2\mathbb{N})^\gamma = (2 \cdot 1)^{\gamma_1} (2 \cdot 2)^{\gamma_2} \cdots (2 \cdot m)^{\gamma_m} \quad \text{if } \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{J}. \tag{2.7}$$

b) *We define the Hida space* $(\mathcal{S})^*$ *of stochastic distributions to be the set of formal expansions*

$$G(\omega) = \sum_{\alpha \in \mathcal{J}} b_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\|G\|_q^2 := \sum_{\alpha \in \mathcal{J}} b_\alpha^2 \alpha! (2\mathbb{N})^{-q\alpha} < \infty \quad \textit{for some } q < \infty. \tag{2.8}$$

We equip $(\mathcal{S})$ with the projective topology and $(\mathcal{S})^*$ with the inductive topology. Convergence in $(\mathcal{S})$ means convergence in $\|\cdot\|_k$ for every $k = 1, 2, \cdots$, while convergence in $(\mathcal{S})^*$ means convergence in $\|\cdot\|_q$ for some $q < \infty$. Then $(\mathcal{S})^*$ can be identified with the dual of $(\mathcal{S})$ and the action of $G \in (\mathcal{S})^*$ on $\psi \in (\mathcal{S})$ is given by

$$\langle G, \psi \rangle_{(\mathcal{S})^*,(\mathcal{S})} := \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha b_\alpha \tag{2.9}$$

In the sequel, we will denote the action $\langle \cdot, \cdot \rangle_{(\mathcal{S})^*,(\mathcal{S})}$ simply with the symbol $\langle \cdot, \cdot \rangle$. We can in a natural way define $(\mathcal{S})^*$-valued integrals as follows:

Definition 2.3 (Integration in $(\mathcal{S})^*$) *Suppose* $Z : \mathbb{R} \to (\mathcal{S})^*$ *has the property that*

$$\langle Z(t), \psi \rangle \in L^1(\mathbb{R}, dt) \qquad \textit{for all } \psi \in (\mathcal{S}).$$

Then the integral

$$\int_{\mathbb{R}} Z(t) dt$$

is defined to be the unique element of $(\mathcal{S})^*$ *such that*

$$\Big\langle \int_{\mathbb{R}} Z(t) dt, \psi \Big\rangle = \int_{\mathbb{R}} \langle Z(t), \psi \rangle dt \qquad \textit{for all } \psi \in (\mathcal{S}). \tag{2.10}$$

Such functions $Z(t)$ *are called* dt-integrable *in* $(\mathcal{S})^*$.

Let $B(t)$ a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. If we consider $B(t)$ as a map $B(\cdot) : \mathbb{R} \to (\mathcal{S})^*$, then $B(t)$ is differentiable with respect to t and its derivative $W(t) := \frac{d}{dt} B(t)$ exists in $(\mathcal{S})^*$ and is called *white noise*.
A fundamental property of the Wick product is the following relation to (Itô-)Skorohod integration. We recall the definition of Skorohod integral.

Let $u(t, \omega)$, $\omega \in \Omega$, $t \in [0, T]$ be a stochastic process (always assumed to be (t, ω)-measurable), such that

$$u(t, \cdot) \qquad \text{is } \mathcal{F}\text{-measurable for all } t \in [0, T] \tag{2.11}$$

and

$$E[u^2(t,\omega)] < \infty \qquad \text{for all } t \in [0,T]. \tag{2.12}$$

Definition 2.4 *Suppose* $u(t,\omega)$ *is a stochastic process satisfying* (2.11), (2.12) *and with Wiener-Itô chaos expansion*

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)). \tag{2.13}$$

Then we define the Skorohod integral *of* u *by*

$$\delta(u) : = \int_{\mathbb{R}} u(t,\omega)\delta B(t) := \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n) \qquad \textit{(when convergent)} \tag{2.14}$$

where $\widetilde{f}_n$ *is the symmetrization of* $f_n(t_1,\dots,t_n,t)$ *as a function of* $n+1$ *variables* $t_1,\dots,t_n,t$.

We say u *is* Skorohod-integrable *and write* $u \in \mathrm{dom}(\delta)$ *if the series in* (2.14) *converges in* $L^2(\mathbb{P})$. *This occurs iff*

$$E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\widetilde{f}_n\|^2_{L^2(\mathbb{R}^{n+1})} < \infty\,. \tag{2.15}$$

Theorem 2.5 *Suppose* $f(t,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ *is Skorohod integrable. Then* $f(t,\cdot) \diamond W(t)$ *is* dt*-integrable in* $(\mathcal{S})^*$ *and*

$$\int_{\mathbb{R}} f(t,\omega)\delta B(t) = \int_{\mathbb{R}} f(t,\omega) \diamond W(t) dt, \tag{2.16}$$

where the integral on the left is the Skorohod integral (which coincides with the Itô integral if f *is adapted) and* $f(t,\omega) \diamond W(t)$ *denotes the Wick product in* $(\mathcal{S})^*$.

2.1 Integration

We now review briefly how the classical white noise theory can be used in order to construct a stochastic integral with respect to a fractional Brownian motion $B^{(H)}(t)$ for any $H \in (0,1)$ as in the approach of [10]. The main idea is to relate the fractional Brownian motion $B^{(H)}(t)$ with Hurst parameter $H \in (0,1)$ to classical Brownian motion $B(t)$ via the following operator M:

Definition 2.6 *The operator $M = M^{(H)}$ is defined on functions $f \in \mathcal{S}(\mathbb{R})$ by*

$$\widehat{Mf}(y) = |y|^{\frac{1}{2}-H}\hat{f}(y); \quad y \in \mathbb{R} \tag{2.17}$$

where

$$\hat{g}(y) := \int_{\mathbb{R}} e^{-ixy} g(x)dx \tag{2.18}$$

denotes the Fourier transform.

For further details on the operator M, we refer to [10] and to [6]. The operator M extends in a natural way from $\mathcal{S}(\mathbb{R})$ to the space

$$\begin{aligned} L^2_H(\mathbb{R}) :=& \{f : \mathbb{R} \to \mathbb{R} \text{ (deterministic)}; \ |y|^{\frac{1}{2}-H}\hat{f}(y) \in L^2(\mathbb{R})\} \\ =& \{f : \mathbb{R} \to \mathbb{R}; Mf(x) \in L^2(\mathbb{R})\} \\ =& \{f : \mathbb{R} \to \mathbb{R}; \|f\|_{L^2_H(\mathbb{R})} < \infty\}, \text{ where } \|f\|_{L^2_H(\mathbb{R})} = \|Mf\|_{L^2(\mathbb{R})}. \end{aligned}$$

The inner product on this space is

$$\langle f, g\rangle_{L^2_H(\mathbb{R})} = \langle Mf, Mg\rangle_{L^2(\mathbb{R})}. \tag{2.19}$$

If $(\xi_n)_{n\in\mathbb{N}}$ is the orthonormal basis of $L^2(\mathbb{R})$ introduced in (2.3), then

$$e_n := M^{-1}\xi_n, \qquad \forall n \in \mathbb{N} \tag{2.20}$$

is an orthonormal basis for $L^2_H(\mathbb{R})$. In particular, the indicator function $\chi_{[0,t]}(\cdot)$ is easily seen to belong to this space, for fixed $t \in \mathbb{R}$, and we write

$$M\chi_{[0,t]}(x) = M[0,t](x).$$

We now define, for $t \in \mathbb{R}$

$$\tilde{B}^{(H)}(t) := \tilde{B}^{(H)}(t,\omega) :=< \omega, M[0,t](\cdot) > \tag{2.21}$$

Then $\tilde{B}^{(H)}(t)$ is Gaussian, $\tilde{B}^{(H)}(0) = E[\tilde{B}^{(H)}(t)] = 0$ for all $t \in \mathbb{R}$ and

$$E\left[\tilde{B}^{(H)}(s)\tilde{B}^{(H)}(t)\right] = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |s-t|^{2H}]$$

as follows by [10], (A.10). Therefore the continuous version of $B^{(H)}(t)$ of $\tilde{B}^{(H)}(t)$ is a fractional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f \in L^2_H(\mathbb{R})$ and define

$$\int_{\mathbb{R}} f(t)dB^{(H)}(t) := \int_{\mathbb{R}} Mf(t)dB(t); \quad f \in L^2_H(\mathbb{R}). \tag{2.22}$$

Now define the *fractional white noise* $W^{(H)}(t)$ as the derivative with respect to t of $B^{(H)}(t)$

$$\frac{dB^{(H)}(t)}{dt} = W^{(H)}(t) \text{ in } (\mathcal{S})^*. \tag{2.23}$$

In particular, by [7] we obtain that the relation between fractional and classical white noise is given by

$$W^{(H)}(t) = MW(t). \tag{2.24}$$

In view of Theorem 2.5 the following definition is natural:

Definition 2.7 (The fractional Wick-Itô-Skorohod (WIS) integral) *Let $Y : \mathbb{R} \to (\mathcal{S})^*$ be such that $Y(t) \diamond W^{(H)}(t)$ is dt-integrable in $(\mathcal{S})^*$. Then we say that Y is $dB^{(H)}$-*integrable *and we define the Wick-Itô-Skorohod (WIS) integral of $Y(t) = Y(t,\omega)$ with respect to $B^{(H)}(t)$ by*

$$\int_{\mathbb{R}} Y(t,\omega)dB^{(H)}(t) := \int_{\mathbb{R}} Y(t) \diamond W^{(H)}(t)dt. \tag{2.25}$$

Note that this definition coincides with (2.22) if $Y = f \in L^2_H(\mathbb{R})$.

Definition 2.8 *A process $Y(t) = \sum_{\alpha\in\mathcal{J}} c_\alpha(t)\mathcal{H}_\alpha(\omega) \in (\mathcal{S})^*$ belongs to the space $\mathcal{M}$ if $c_\alpha(\cdot) \in L^2_H(\mathbb{R})$ and $\sum_{\alpha\in\mathcal{J}} Mc_\alpha(t)\mathcal{H}_\alpha(\omega)$ converges in $(\mathcal{S})^*$ for all t.*

Then the following fundamental relation holds.

Proposition 2.9 (Integration)[BØSW, (5.2)], [Ø, (3.16)] *Suppose $Y : \mathbb{R} \to (\mathcal{S})^*$ is $dB^{(H)}$-integrable (Definition 2.7) and $Y \in \mathcal{M}$. Then*

$$\int_{\mathbb{R}} Y(t)dB^{(H)}(t) = \int_{\mathbb{R}} MY(t)\delta B(t). \tag{2.26}$$

2.2 Differentiation

We now recall the approach in [16] to differentiation, as modified and extended by [10]:

Definition 2.10 *Let $F : \Omega \to \mathbb{R}$ and choose $\gamma \in \Omega$. Then we say F has a directional M-derivative in the direction γ if*

$$D^{(H)}_\gamma F(\omega) := \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}[F(\omega + \varepsilon M\gamma) - F(\omega)] \tag{2.27}$$

exists almost surely in $(\mathcal{S})^*$. In that case we call $D^{(H)}_\gamma F$ the directional M-derivative of F in the direction γ.

Definition 2.11 *We say that $F : \Omega \to \mathbb{R}$ is differentiable if there exists a function*

$$\Psi : \mathbb{R} \to (\mathcal{S})^*$$

in $\mathcal{M}$ such that

$$D_\gamma^{(H)} F(\omega) = \int_{\mathbb{R}} M\Psi(t) M\gamma(t) dt \qquad \text{for all} \quad \gamma \in L_H^2(\mathbb{R}). \tag{2.28}$$

Then we write

$$D_t^{(H)} F := \frac{\partial^{(H)}}{\partial \omega} F(t, \omega) = \Psi(t) \tag{2.29}$$

and we call $D_t^{(H)} F$ the Malliavin derivative or the stochastic gradient of F at t.

In the classical case ($H = \frac{1}{2}$) we use the notation D_t for the corresponding Malliavin derivative.

Proposition 2.12 [BØSW, (5.1)] *Let $F \in (\mathcal{S})^*$. Then*

$$D_t F = M D_t^{(H)} F \quad \text{for a.a. } t \in \mathbb{R}. \tag{2.30}$$

Proposition 2.13 [BØSW, Theorem 5.3] *Suppose $Y : \mathbb{R} \to (\mathcal{S})^*$ is $dB^{(H)}$-integrable. If $D_t Y(\cdot) : \mathbb{R} \to (\mathcal{S})^*$ is $dB^{(H)}$-integrable for every t, then*

$$D_t^{(H)} (\int_{\mathbb{R}} Y(s) dB^{(H)}(s)) = \int_{\mathbb{R}} D_t^{(H)} Y(s) dB^{(H)}(s) + Y(t). \tag{2.31}$$

Definition 2.14 *Let $\mathbb{D}_{1,2}^{(H)}$ be the set of all $F \in L^2(\mathbb{P})$ such that the Malliavin derivative $D_t^{(H)} F$ exists and*

$$E\left[\int_{\mathbb{R}} [D_t^{(H)} F]^2 dt\right] < \infty \tag{2.32}$$

The following result has been obtained with a different proof in Lemma 2 of [18].

Lemma 2.15 *Suppose $g \in L_H^2(\mathbb{R})$ and let $F \in \mathbb{D}_{1,2}^{(H)}$. Then*

$$F \diamond \int_{\mathbb{R}} g(t) dB^{(H)}(t) = F \cdot \int_{\mathbb{R}} g(t) dB^{(H)}(t) - \langle g, D_\cdot^{(H)} F \rangle_{L_H^2(\mathbb{R})} \tag{2.33}$$

3 The forward integral

By following the approach of [23], we now define the *forward integral* with respect to the fractional Brownian motion as follows:

Definition 3.1

a) The (classical) forward integral *of a real valued measurable process* Y *with integrable trajectories is defined by*

$$\int_0^T Y(t)d^-B^{(H)}(t) = \lim_{\epsilon \longrightarrow 0} \int_0^T Y(t)\frac{B^{(H)}(t+\epsilon) - B^{(H)}(t)}{\epsilon}dt,$$

provided that the limit exists in probability under $\mathbb{P}$.

b) The (generalized) forward integral *of a real valued measurable process* Y *with integrable trajectories is defined by*

$$\int_0^T Y(t)d^-B^{(H)}(t) = \lim_{\epsilon \longrightarrow 0} \int_0^T Y(t)\frac{B^{(H)}(t+\epsilon) - B^{(H)}(t)}{\epsilon}dt,$$

provided that the limit exists in $(\mathcal{S})^*$.

Note that in the generalized definition of forward integral, the limit is required to exist in the *Hida space of stochastic distributions* $(\mathcal{S})^*$ introduced in Definition 2.2. Convergence in $(\mathcal{S})^*$ is also explained in Section 2.

Corollary 3.2 *Let* $\psi(t) = \psi(t, \omega)$ *be a measurable forward integrable process and assume that* ψ *is càglàd.The forward integral of* ψ *with respect to the fractional Brownian motion* $B^{(H)}$ *coincides with*

$$\int_0^T \psi(t)d^-B^{(H)}(t) = \lim_{|\Delta| \longrightarrow 0} \sum_{j=1}^{N} \psi(t_j)\Delta B_{t_j}^{(H)} \tag{3.1}$$

whenever the left-hand limit exists in probability, where $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ *is a partition of* $[0, T]$ *with mesh size* $|\Delta| = \sup_{j=0,\cdots,N-1} |t_{j+1} - t_j|$ *and* $\Delta B_{t_j}^{(H)} = B_{t_{j+1}}^{(H)} - B_{t_j}^{(H)}$.

Proof. Let ψ be a càglàd forward integrable process and

$$\psi^{(\Delta)}(t) = \sum_k \psi(t_k)\chi_{(t_k, t_{k+1}]}(t) \tag{3.2}$$

be a càglàd step function approximation to ψ. Then $\psi^{(\Delta)}(t)$ converges boundedly almost surely to $\psi(t)$ as $|\Delta| \longrightarrow 0$. The forward integral of $\psi^{(\Delta)}(t)$ is then given by

$$\begin{aligned}
\int_0^T \psi^{(\Delta)}(t) d^- B^{(H)}(t) &= \lim_{\epsilon \to 0} \int_0^T \psi^{(\Delta)}(s) \frac{B^{(H)}(s+\epsilon) - B^{(H)}(s)}{\epsilon} ds \\
&= \lim_{\epsilon \to 0} \sum_k \psi(t_k) \int_{t_k}^{t_{k+1}} \frac{1}{\epsilon} \int_s^{s+\epsilon} dB^{(H)}(u) ds \\
&= \lim_{\epsilon \to 0} \sum_k \psi(t_k) \int_{t_k}^{t_{k+1}} \frac{1}{\epsilon} \int_{u-\epsilon}^{u} ds dB^{(H)}(u) \\
&= \sum_k \psi(t_k) \Delta B_{t_k}^{(H)}, \qquad (3.3)
\end{aligned}$$

where $\Delta B_{t_k}^{(H)} = B_{t_{k+1}}^{(H)} - B_{t_k}^{(H)}$. Hence (3.1) follows by the dominated convergence theorem and by (3.3). $\square$

For the sequel we will use the same notation as in Section 2.

Definition 3.3 *The space $\mathbb{L}_{1,2}^{(H)}$ consists of all càglàd processes*

$$\psi(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) \mathcal{H}_\alpha(\omega) \in (\mathcal{S})^*$$

for every $t \in [0, T]$ and such that

$$\|\psi\|_{\mathbb{L}_{1,2}^{(H)}}^2 := \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \alpha_i \alpha! \|c_\alpha\|_{L^2([0,T])}^2 < \infty. \qquad (3.4)$$

Note that if $\psi(t) \in (\mathcal{S})^*$ for every $t \in [0, T]$, then $D_s \psi(t)$ exists in $(\mathcal{S})^*$ (see Lemma 3.10 of [1]). We recall a preliminary lemma needed in the following.

Lemma 3.4 *Let $(\Gamma, \mathcal{G}, m)$ be a measure space. Let $f_\epsilon : \Gamma \to B$, $\epsilon \in \mathbb{R}$, be measurable functions with values in a Banach space $(B, \|\cdot\|_B)$. If $f_\epsilon(\gamma) \to f_0(\gamma)$ as $\epsilon \to 0$ for almost every $\gamma \in \Gamma$ and there exists $K < \infty$ such that*

$$\int_\Gamma \|f_\epsilon(\gamma)\|_B^2 dm(\gamma) < K \qquad (3.5)$$

for all $\epsilon \in \mathbb{R}$, then

$$\int_\Gamma f_\epsilon(\gamma) dm(\gamma) \to \int_\Gamma f_0(\gamma) dm(\gamma) \qquad (3.6)$$

in $\|\cdot\|_B$.

PROOF. The proof is analogous to the one of Theorem II.21.2 of [22]. $\square$

Lemma 3.5 *Suppose that* $\psi \in \mathbb{L}_{1,2}^{(H)}$. *Then*

$$M_{t+}D_{t+}\psi(t) := \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds \tag{3.7}$$

exists in $L^2(\mathbb{P})$ *for all* t. *Moreover*

$$\int_0^T M_{t+}D_{t+}\psi(t)dt = \lim_{\epsilon \longrightarrow 0} \int_0^T \left(\frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds \right) dt \tag{3.8}$$

in $L^2(\mathbb{P})$ *and*

$$E\left[\left(\int_0^T M_{s+}D_{s+}\psi(s)ds \right)^2 \right] < \infty. \tag{3.9}$$

PROOF. Suppose that $\psi(t)$ has the expansion

$$\psi(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) \mathcal{H}_\alpha(\omega).$$

In the sequel we drop ω in $\mathcal{H}_\alpha(\omega)$ for the sake of simplicity. Then we have

$$D_s\psi(t) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha(t) \alpha_i \mathcal{H}_{\alpha - \epsilon^{(i)}} \xi_i(s)$$

and

$$M_s D_s \psi(t) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha(t) \alpha_i \mathcal{H}_{\alpha - \epsilon^{(i)}} \eta_i(s),$$

where $\eta_i(s) = M\xi_i(s)$. Hence

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} (c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds) \alpha_i \mathcal{H}_{\alpha - \epsilon^{(i)}} .$$

Since $\eta_i(s) = M\xi(s)$ is a continuous function, we have that

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \to \eta_i(t)$$

as $\epsilon \to 0$.

We apply now Lemma 3.4 with $\gamma = (\alpha, i)$, $dm(\gamma) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \delta_{(\alpha,i)}$, where

δ_x denotes the point mass at x, $B = L^2(\mathbb{P})$ and $f_\epsilon = (c_\alpha(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds)\alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}$.
We obtain

$$\begin{aligned}\int_\Gamma \|f_\epsilon(\gamma)\|_B^2 dm(\gamma) &= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^\infty \|f_\epsilon(\gamma)\|_{L^2(\mathbb{P})}^2\\ &= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}(c_\alpha(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds)^2\alpha_i\alpha!\\ &\le \left[\frac{(t+\epsilon)^{2H}-t^{2H}}{\epsilon}\right]^2\sum_{\alpha\in\mathcal{J}}\sum_{i=1}c_\alpha(t)^2\alpha_i\alpha!\,,\end{aligned}$$

since

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds = \langle M\xi_i, \frac{1}{\epsilon}\chi_{[t,t+\epsilon]}\rangle_{L^2(\mathbb{R})} =$$
$$\langle M^2 e_i, \frac{1}{\epsilon}\chi_{[t,t+\epsilon]}\rangle_{L^2(\mathbb{R})} = \langle e_i, \frac{1}{\epsilon}\chi_{[t,t+\epsilon]}\rangle_{L^2_H(\mathbb{R})} \le$$
$$\|e_i\|_{L^2_H(\mathbb{R})}\ \frac{1}{\epsilon}\|\chi_{[t,t+\epsilon]}\|_{L^2_H(\mathbb{R})} = \frac{(t+\epsilon)^{2H}-t^{2H}}{\epsilon},$$

where we have used that the fact that $\|e_i\|_{L^2_H(\mathbb{R})} = 1$ and the equality

$$\int_\mathbb{R}[M[a,b](x)]^2dx = (b-a)^{2H}.$$

Since we have $\sum_{\alpha\in\mathcal{J}}\sum_{i=1}c_\alpha(t)^2\alpha_i\alpha! < \infty$ for almost every t, by Lemma 3.4 it follows that $\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^\infty(c_\alpha(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds)\alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}$ converges to

$$\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^\infty c_\alpha(t)\eta_i(t)\alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}$$

in $L^2(\mathbb{P})$.
We now prove (3.8). Consider

$$\int_0^T\frac{1}{\epsilon}\int_t^{t+\epsilon}M_sD_s\psi(t)dsdt = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^\infty\int_0^T\left(c_\alpha(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds\right)dt\ \alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}\,.$$

Now assuming $f_\epsilon = \int_0^T\left(c_\alpha(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds\right)dt\ \alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}$ and as before $\gamma = (\alpha, i)$, $B = L^2(\mathbb{P})$, $dm(\gamma) = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^\infty\delta_{\alpha,i}$, where δ_x denotes the point mass at x, we use again Lemma 3.4. We obtain

$$
\begin{aligned}
\int_\Gamma \|f_\epsilon(\gamma)\|_B^2 dm(\gamma) &= \sum_{\alpha\in\mathcal{J}} \sum_{i=1}^{\infty} \|f_\epsilon(\gamma)\|_{L^2(\mathbb{P})}^2 \\
&= \sum_{\alpha\in\mathcal{J}} \sum_{i=1} \left(\int_0^T c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \ dt \right)^2 \ \alpha_i \alpha! \\
&\le \sum_{\alpha\in\mathcal{J}} \sum_{i=1} \left(\int_0^T c_\alpha(t) \left[\frac{(t+\epsilon)^{2H} - t^{2H}}{\epsilon} \right] dt \right)^2 \ \alpha_i \alpha! \\
&\le \sum_{\alpha\in\mathcal{J}} \sum_{i=1} (\int_0^T c_\alpha(t)^2 dt) (\int_0^T \left[\frac{(t+\epsilon)^{2H} - t^{2H}}{\epsilon} \right]^2 dt) \ \alpha_i \alpha! \,.
\end{aligned}
\tag{3.10}
$$

Since $\psi \in \mathbb{L}_{1,2}^{(H)}$ by Lemma 3.4 we can conclude that the limit 3.8 exists in $L^2(\mathbb{P})$ and also that (3.9) holds. □

Lemma 3.6 *Suppose that $\psi \in \mathbb{L}_{1,2}^{(H)}$ and let*

$$\psi^{(\Delta)}(s) = \sum_k \psi(t_k) \chi_{(t_k, t_{k+1}]}(s) \tag{3.11}$$

be a càglàd step function approximation to ψ, where $\Delta = \max_i |\Delta t_i|$ is the maximal length of the subinterval in the partition $0 = t_0 < \cdots < t_n = T$ of $[0, T]$. Then $\psi^{(\Delta)} \in \mathbb{L}_{1,2}^{(H)}$ for all Δ and

$$\int_0^T M_{s+} D_{s+} \psi^{(\Delta)}(s) ds \longrightarrow \int_0^T M_{s+} D_{s+} \psi(s) ds \qquad \text{in } L^2(\mathbb{P}) \tag{3.12}$$

as $|\Delta| \longrightarrow 0$.

Proof. Since $\psi^{(\Delta)}(s) = \sum_{\alpha\in\mathcal{J}} c_\alpha^{(\Delta)}(s) \mathcal{H}_\alpha(\omega)$ with

$$c_\alpha^{(\Delta)}(s) = \sum_k c_\alpha(t_k) \chi_{(t_k, t_{k+1}]}(s)$$

and

$$\|c_\alpha^{(\Delta)}\|_{L^2([0,T])} \le const. \|c_\alpha\|_{L^2([0,T])} \quad \forall \alpha, \tag{3.13}$$

it follows that $\psi^{(\Delta)} \in \mathbb{L}_{1,2}^{(H)}$. We have

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s D_s \psi^{(\Delta)}(t)ds = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\left(\int_0^T (c_\alpha^{(\Delta)}(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds)dt\right)\alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}\,.$$

If we assume $\gamma = (\alpha, i)$, $B = L^2(\mathbb{P})$, $m(d\gamma) = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\delta_{(\alpha,i)}$, where δ_x denotes the point mass at x, and $f_\Delta = \left(\int_0^T c_\alpha^{(\Delta)}(t)\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds)dt\right)\alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}}$, with the same argument as in (3.10) by Lemma 3.4 we obtain that

$$\int_0^T\left(\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s D_s\psi(t)ds\right)dt = \lim_{|\Delta|\longrightarrow 0}\int_0^T\left(\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s D_s\psi^{(\Delta)}(t)ds\right)dt \tag{3.14}$$

in $L^2(\mathbb{P})$ for almost every s, since $c_\alpha^{(\Delta)}$ converges by dominated convergence to c_α in $L^2(\mathbb{P})$ and $\psi^{(\Delta)} \in \mathbb{L}_{1,2}^{(H)}$. Using (3.14) and Lemma 3.5 we conclude that (3.12) holds. □

We now investigate the relation among forward integrals and WIS-integrals for $H > \frac{1}{2}$.

In [4] and [19] a similar relation is established between the symmetric integral and the divergence, in [9] between the forward integral and the fractional Wick-Itô-Skorohod integral. For the case $H < \frac{1}{2}$, we refer to [2].

Theorem 3.7 *Let $H \in (0,1)$. Suppose $\psi \in \mathbb{L}_{1,2}^{(H)}$ and that one of the following conditions holds:*

i) ψ is Wick-Itô-Skorohod integrable (Definition 2.7);

ii) ψ is forward integrable in $(\mathcal{S})^*$ (Definition 3.1).

Then

$$\int_0^T \psi(t)d^-B^{(H)}(t) = \int_0^T \psi(t)dB^{(H)}(t) + \int_0^T M_{t+}D_{t+}\psi(t)dt, \tag{3.15}$$

holds as an identity in $(\mathcal{S})^*$, where here $\int_0^T \psi(t)dB^{(H)}(t)$ is the WIS-integral of Definition 2.7.

Proof. We prove (3.15) assuming that hypothesis $i)$ is in force. The argument works symmetrically under hypothesis $ii)$. Let $\psi \in \mathbb{L}_{1,2}^{(H)}$. Since ψ is càglàd, we can approximate it as

$$\psi(t) = \lim_{|\Delta t|\longrightarrow 0}\sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t) \qquad \text{a.e.}$$

where for any partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$, with $\Delta t_j = t_{j+1} - t_j$, we have put $|\Delta t| = \sup_{j=0,\cdots,N-1} \Delta t_j$.

As before we put $\psi^{(\Delta)}(t) = \sum_{j=0}^{N-1} \psi(t_k)\chi_{(t_k,t_{k+1}]}(t)$ and evaluate

$$\int_0^T \psi^{(\Delta)}(t)d^- B^{(H)}(t) = \lim_{\epsilon\to 0}\int_0^T \psi^{(\Delta)}(t,\omega)\frac{B^{(H)}(t+\epsilon) - B^{(H)}(t)}{\epsilon}dt =$$

$$\lim_{\epsilon\to 0}\int_0^T (\sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t))\frac{1}{\epsilon}\int_t^{t+\epsilon} dB^{(H)}(u)dt =$$

$$\lim_{\epsilon\to 0}\int_0^T (\sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t))\diamond\frac{1}{\epsilon}\int_t^{t+\epsilon} dB^{(H)}(u)dt+$$

$$\lim_{\epsilon\to 0}\sum_j\int_0^T \chi_{(t_j,t_{j+1}]}(t)\frac{1}{\epsilon}\int_{\mathbb{R}}\chi_{[t,t+\epsilon]}(u)M_u^2 D_u^{(H)}\psi(t_j)dudt\,.$$

The first limit is equal to

$$\lim_{\epsilon\to 0}\int_0^T (\sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t))\diamond\frac{1}{\epsilon}\int_t^{t+\epsilon} dB^{(H)}(u)dt =$$

$$\lim_{\epsilon\to 0}\int_0^T (\sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t))\diamond\frac{1}{\epsilon}\int_t^{t+\epsilon} W^{(H)}(u)dudt =$$

$$\lim_{\epsilon\to 0}\int_0^T \frac{1}{\epsilon}(\int_{u-\epsilon}^u \sum_j \psi(t_j)\chi_{(t_j,t_{j+1}]}(t))\diamond W^{(H)}(u)du =$$

$$\int_0^T \psi^{(\Delta)}(u)\diamond W^{(H)}(u)du,$$

that converges in $(\mathcal{S})^*$ to $\int_0^T \psi(u)\diamond W^{(H)}(u)du = \int_0^T \psi(u)dB^{(H)}(u)$. For the second limit we get

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\sum_j\int_0^T \chi_{(t_j,t_{j+1}]}(t)\int_t^{t+\epsilon} M_u^2 D_u^{(H)}\psi(t_j)dudt =$$

$$\lim_{\epsilon\to 0}\int_0^T \frac{1}{\epsilon}\int_t^{t+\epsilon} M_u^2 D_u^{(H)}\psi^{(\Delta)}(t)dudt =$$

$$\lim_{\epsilon\longrightarrow 0}\int_0^T \frac{1}{\epsilon}\int_t^{t+\epsilon} M_u D_u\psi^{(\Delta)}(t)dudt.$$

By Lemmas 3.5 and 3.6 the last limit converges to

$$\int_0^T M_{u+}D_{u+}\psi(u)du \tag{3.16}$$

in $L^2(\mathbb{P})$. $\square$

An analogous relation to the one of Theorem 3.7 between Stratonovich integrals and Wick-Itô-Skorohod integrals for fractional Brownian motion is proved under different conditions in [18].
An Itô formula for forward integrals with respect to classical Brownian motion was obtained by [23] and then extended to the fractional Brownian motion case in [12]. Here we prove the following Itô formula for forward integrals with respect to fractional Brownian motion as a consequence of Lemma 3.8.

Lemma 3.8 *Let $G \in (\mathcal{S})^*$ and suppose that ψ is forward integrable. Then*

$$G(\omega) \int_0^T \psi(t) d^- B^{(H)}(t) = \int_0^T G(\omega)\psi(t) d^- B^{(H)}(t) \tag{3.17}$$

PROOF. This is immediate by Definition 3.1. $\square$

Definition 3.9 *Let ψ be a forward integrable process and let $\alpha(s)$ be a measurable process such that $\int_0^t |\alpha(s)|ds < \infty$ a.s. for all $t \geq 0$. Then the process*

$$X(t) := x + \int_0^t \alpha(s)ds + \int_0^t \psi(s)d^- B^{(H)}(s); \quad t \geq 0 \tag{3.18}$$

is called a fractional forward process. *As a shorthand notation for (3.18) we write*

$$d^- X(t) := \alpha(t)dt + \psi(t)d^- B^{(H)}(t); \quad X(0) = x. \tag{3.19}$$

Theorem 3.10 *Let*

$$d^- X(t) = \alpha(t)dt + \psi(t)d^- B^{(H)}(t); \; X(0) = x$$

be a fractional forward process. Suppose $f \in C^2(\mathbb{R}^2)$ and put $Y(t) = f(t, X(t))$. Then if $\frac{1}{2} < H < 1$, we have

$$d^- Y(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))d^- X(t)$$

PROOF. Let $0 = t_0 < t_1 < \cdots < t_N = t$ be a partition of $[0, t]$. By using Taylor expansion, we get by equation (3.17)

$$
\begin{aligned}
Y(t) - Y(0) &= \sum_j Y(t_{j+1}) - Y(t_j) \\
&= \sum_j f(t_{j+1}, X(t_{j+1})) - f(t_j, X(t_j)) \\
&= \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j))\Delta t_j + \sum_j \frac{\partial f}{\partial x}(t_j, X(t_j))\Delta X(t_j) \\
&+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))(\Delta X(t_j))^2 + \sum_j o((\Delta t_j)^2) + o((\Delta X(t_j))^2) \\
&= \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j))\Delta t_j + \sum_j \int_{t_j}^{t_{j+1}} \frac{\partial f}{\partial x}(t_j, X(t_j))d^- X_t \\
&+ \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))(\Delta X(t_j))^2 + \sum_j o((\Delta t_j)^2) + o((\Delta X(t_j))^2)
\end{aligned}
$$

where $\Delta X(t_j) = X(t_{j+1}) - X(t_j)$. Since $\frac{1}{2} < H < 1$, the quadratic variation of the fractional Brownian motion is zero and we are left with the first terms of the sum above, which converges to $\int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, X(s))d^- X(s)$. □

Using the results of Theorem 3.7 and 3.10, we obtain a general Itô formula for functionals of Wick-Itô-Skorohod integrals with respect to the fractional Brownian motion when $\frac{1}{2} < H < 1$. An Itô formula for $\frac{1}{2} < H < 1$ has been already proved in [9] and in [4], but under more restrictive hypotheses. Here we provide a different proof under weaker assumptions. If $\frac{1}{2} < H < 1$ this theorem extends Theorem 3.8 in [7]. A related result, obtained independently and by a different method, can be found in [11]. Moreover our results hold in a different setting.

Theorem 3.11 (Itô formula for the WIS-integral) *Suppose* $\frac{1}{2} < H < 1$. *Let* $\gamma(s)$ *be a measurable process such that* $\int_0^t |\gamma(s)|ds < \infty$ *a.s. for all* $t \geq 0$, *let* $\psi(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t)\mathcal{H}_\alpha(\omega)$ *be càglàd, WIS-integrable and such that*

$$\sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \|c_\alpha\|_{L^2([0,T])} \alpha_i (\alpha_k + 1)\alpha! < \infty.$$

Suppose that $M_tD_t\psi(s)$ is also WIS-integrable for almost all $t \in [0,T]$. Consider

$$X(t) = x + \int_0^t \gamma(s)ds + \int_0^t \psi(s)dB^{(H)}(s), \quad t \in [0,T],$$

or, in short-hand notation,

$$dX(t) = \gamma(t)dt + \psi(t)dB^{(H)}(t), \quad X(0) = x.$$

Suppose X_t has a càdlàg version (Remark 3.12). Let $f \in C^2(\mathbb{R}^2)$ and put $Y(t) = f(t, X(t))$. Then on $[0,T]$

$$dY(t) = \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))dX(t) + \frac{\partial^2 f}{\partial x^2}(t,X(t))\psi(t)M_{t+}D_{t+}X(t)dt, \tag{3.20}$$

and equivalently

$$\begin{aligned} dY(t) = {} & \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))dX(t) + \frac{\partial^2 f}{\partial x^2}(t,X(t))\psi(t)M^2(\psi\chi_{[0,t]})_t dt \\ & + \left[\frac{\partial^2 f}{\partial x^2}(t,X(t))\psi(t)\int_0^t M_t^2 D_t^{(H)}\psi(u)dB^{(H)}(u)\right] dt, \end{aligned} \tag{3.21}$$

where $M^2(\psi\chi_{[0,t]})_t = M^2(\psi\chi_{[0,t]})(t)$.

Proof. For simplicity we put $\alpha = 0$. By Theorem 3.7 we have

$$X(t) = \int_0^t \psi(s)d^-B^{(H)}(s) - \int_0^t M_{s+}^2 D_{s+}^{(H)}\psi(s)ds$$

We note that

$$\begin{aligned} \frac{1}{\epsilon}\int_t^{t+\epsilon} M_s^2 D_s^{(H)}(f'(X(t))\psi(t))ds = {} & f'(X(t))\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s^2 D_s^{(H)}\psi(t)ds \\ & + \psi(t)f''(X(t))\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s^2 D_s^{(H)}X(t)ds \end{aligned} \tag{3.22}$$

Since $\psi \in \mathbb{L}_{1,2}^{(H)}$, the first term converges to $f'(X(t))M_{t+}^2 D_{t+}^{(H)}\psi(t)$ as $\epsilon \to 0$. For the second term we restrict our attention to

$$\begin{aligned} \frac{1}{\epsilon}\int_t^{t+\epsilon} M_s^2 D_s^{(H)}X(t)ds = {} & \underbrace{\frac{1}{\epsilon}\int_t^{t+\epsilon}\int_0^t M_s^2 D_s^{(H)}\psi(u)dB^{(H)}(u)ds}_{a)} \\ & + \underbrace{\frac{1}{\epsilon}\int_t^{t+\epsilon} M_s^2(\psi\chi_{[0,t]})ds}_{b)}. \end{aligned}$$

a) To study the convergence of the term a), we proceed as in Lemma 3.5. By using the chaos expansion we obtain

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}\int_0^t M_s^2 D_s^{(H)}\psi(u)dB^{(H)}(u)ds = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(c_\alpha,\xi_k)_t\ \frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds\ \alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}+\epsilon^{(k)}}.$$

Put $\psi_{i,k,\alpha,\epsilon} := (c_\alpha,\xi_k)_t\ \frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds\ \alpha_i\mathcal{H}_{\alpha-\epsilon^{(i)}+\epsilon^{(k)}}$. Then

$$\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\|\psi_{i,k,\alpha,\epsilon}\|^2_{L^2(\mathbb{P})} = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(c_\alpha,\xi_k)_t^2\left(\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds\right)^2\alpha_i(\alpha_k+1)\alpha! \le$$
$$\left[\frac{(t+\epsilon)^{2H}-t^{2H}}{\epsilon}\right]^2\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\|c_\alpha\|^2_{L^2(0,T)}\|\xi_k\|^2_{L^2(0,T)}\alpha_i(\alpha_k+1)\alpha! \le$$
$$\left[\frac{(t+\epsilon)^{2H}-t^{2H}}{\epsilon}\right]^2\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\|c_\alpha\|^2_{L^2(0,T)}\alpha_i(\alpha_k+1)\alpha!\,, \tag{3.23}$$

where we have used that $\|\xi_k\|^2_{L^2(0,T)} \le \|\xi_k\|^2_{L^2(0,T)} = 1$, $\forall k = 1,2,\cdots$. Since

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}\eta_i(s)ds \to \eta_i(t) \tag{3.24}$$

and (3.23) holds, by Lemma 3.4 we conclude that

$$\lim_{\epsilon\longrightarrow 0}\frac{1}{\epsilon}\int_t^{t+\epsilon}\int_0^t M_s^2 D_s^{(H)}\psi(u)dB^{(H)}(u)ds = \int_0^t M_t^2 D_t^{(H)}\psi(u)dB^{(H)}(u) \tag{3.25}$$

in $L^2(\mathbb{P})$.

b) Since $\psi\in\mathbb{L}_{1,2}^{(H)}$, we have

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}M_s^2(\psi\chi_{[0,t]})ds \longrightarrow M^2(\psi\chi_{[0,t]})_t, \quad \text{a.e. and in } L^2(\mathbb{P}), \tag{3.26}$$

where for the sake of simplicity we have put $M^2(\psi\chi_{[0,t]})_t = M^2(\psi\chi_{[0,t]})(t)$. Let $A_t = -\int_0^t M_{s+}^2 D_{s+}^{(H)}\psi(s)ds$. Then by the Itô formula for forward integrals

(Theorem 3.10) we obtain

$$\begin{aligned}
dY(t) &= f'(X(t))d^{-}X(t) \\
&= f'(X(t))dA_t + f'(X(t))d^{-}B^{(H)}(t) \\
&= -f'(X(t))M_{t+}D_{t+}\psi(t)dt + f'(X(t))\psi(t)dB^{(H)}(t) \\
&+ \left[f'(X(t))M_{t+}D_{t+}\psi(t) + \psi(t)f''(X(t))M_{t+}D_{t+}X(t)\right] dt \\
&= f'(X(t))dX(t) + f''(X(t))\psi(t)M_{t+}D_{t+}X(t)dt
\end{aligned}$$

and by (3.25) and (3.26) we can conclude that

$$\begin{aligned}
dY(t) = f'(X(t))dX(t) + f''(X(t))\psi(t)\int_0^t M_t^2 D_t^{(H)}\psi(u)dB^{(H)}(u)dt \\
+ f''(X(t))\psi(t)M^2(\psi\chi_{[0,t]})_t dt.
\end{aligned}$$

Note that all the integrands appearing in (3.27) are well-defined because X_t is càdlàg. $\square$

Remark 3.12 *Conditions under which the integral process admits a continuous modification are proved in [3] and [4].*

Corollary 3.13 *Assume that* $\psi \in L^2_H(\mathbb{R})$*,* $\alpha = 0$ *and otherwise let* H, X, f, Y *be as in Theorem 3.11. Then*

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX(t) + \frac{\partial^2 f}{\partial x^2}(t, X(t))\psi(t)M^2(\chi_{[0,t]}\psi)_t dt \tag{3.27}$$

Remark 3.14 *In the case when* $\psi(s)$ *is* deterministic*, a (different) Itô formula, valid for all* $H \in (0,1)$ *and for all* x*-entire functions* $f(t,x)$ *of order* 2*, has been obtained in Theorem 11.1 of [15].*

4 Examples

4.1 A special case

In [5] and [7] an Itô formula for the case when $Y(t) = f(B^{(H)}(t))$ is provided, valid for all $H \in (0,1)$. We recall here that formula

$$dY(t) = f'(X(t))dX(t) + Ht^{2H-1}f''(X(t))\psi(t)dt \tag{4.1}$$

We now show that if $H > \dfrac{1}{2}$ then (3.20) and (4.1) coincide in this case.

Proposition 4.1 *For every $H \in (0,1)$ we have*

$$M_{t+}D_{t+}B^{(H)}(t) = Ht^{2H-1}, \quad t \geq 0.$$

Proof. Let $t \geq 0$. We recall that $D_t^{(H)}B^{(H)}(u) = \chi_{[0,u)}(t)$. Hence we need to prove that

$$\begin{aligned} M_{t+}D_{t+}B^{(H)}(t) &= \lim_{s \longrightarrow t^+} \frac{1}{\epsilon} \int_t^{t+\epsilon} M_s^2 D_s^{(H)} B^{(H)}(t) ds \\ &= [M_t^2 \chi_{[0,u)}(t)]_{u=t} = Ht^{2H-1} \end{aligned}$$

We consider $\psi(u) = \int_{\mathbb{R}} (M_t \chi_{[0,u)}(t))^2 dt$. Since, by [10], we have that $\psi(u) = u^{2H}$, we only need to show that $\psi'(u) = 2[M_t^2 \chi_{[0,u)}(t)]_{t=u}$. We rewrite $\psi(u)$ as follows

$$\begin{aligned} \psi(u) &= \int_{\mathbb{R}} (M_t \chi_{[0,u)}(t))^2 dt \\ &= \int_{\mathbb{R}} \chi_{[0,u)}(t) M_t^2 \chi_{[0,u)}(t) dt \\ &= \int_0^u M_t^2 \chi_{[0,u)}(t) dt \end{aligned}$$

by using the properties of the operator M. We compute

$$\begin{aligned} &\frac{\psi(u+\epsilon) - \psi(u)}{\epsilon} \\ &= \frac{1}{\epsilon}\left(\int_0^{u+\epsilon} M_t^2 \chi_{[0,u+\epsilon]}(t) dt - \int_0^u M_t^2 \chi_{[0,u)}(t) dt \right) \\ &= \frac{1}{\epsilon}\left(\int_u^{u+\epsilon} M_t^2 \chi_{[0,u+\epsilon]}(t) dt + \int_0^u [M_t^2 \chi_{[0,u+\epsilon]}(t) - M_t^2 \chi_{[0,u)}(t)] dt \right) \end{aligned}$$

by adding and subtracting $\int_0^u M_t^2 \chi_{[0,u+\epsilon]}(t) dt$. Since the operator M transforms $\chi_{[0,u)}(t)$ into a continuous function, we obtain

1. $\int_u^{u+\epsilon} M_t^2 \chi_{[0,u+\epsilon]}(t) dt = [M_t^2 \chi_{[0,u+\epsilon]}(t)]_{t=\xi_\epsilon} \epsilon$, where $u < \xi_\epsilon < u+\epsilon$. By writing

   $$[M_t^2 \chi_{[0,u+\epsilon]}(t)]_{t=\xi_\epsilon} = [M_t^2 (\chi_{[0,u+\epsilon]} - \chi_{[0,u)})(t)]_{t=\xi_\epsilon} + [M_t^2 \chi_{[0,u)}(t)]_{t=\xi_\epsilon}$$

   we obtain that, when taking the limit as $\epsilon \longrightarrow 0$, the first term goes to zero, while the second term converges to $[M_t^2 \chi_{[0,u)}(t)]_{t=u}$ since $\xi_\epsilon \longrightarrow u$ when $\epsilon \longrightarrow 0$.

2. We have that

$$\frac{1}{\epsilon}\int_0^u [M_t^2\chi_{[0,u+\epsilon]}(t)dt - M_t^2\chi_{[0,u)}(t)]dt =$$
$$\frac{1}{\epsilon}\int_0^u M_t^2[\chi_{(u,u+\epsilon]}(t)]dt =$$
$$\frac{1}{\epsilon}\int_0^T \chi_{[0,u)}(t)(M_t^2[\chi_{(u,u+\epsilon]}(t)]dt =$$
$$\frac{1}{\epsilon}\int_u^{u+\epsilon} M_t^2[\chi_{[0,u)}(t)]dt$$

converges to $[M_t^2\chi_{[0,u)}(t)]_{t=u}$ as $\epsilon \longrightarrow 0$.

Hence

$$\psi'(u) = \lim_{\epsilon\longrightarrow 0}\frac{\psi(u+\epsilon)-\psi(u)}{\epsilon} = 2[M_t^2\chi_{[0,u)}(t)]_{t=u}$$

i.e. the equality $[M_t^2\chi_{[0,u)}(t)]_{t=u} = Hu^{2H-1}$ holds for every $H \in (0,1)$. □

4.2 An integration by parts formula

Let $\psi(s) = \psi(s,\omega) \in \mathbb{L}_{1,2}^{(H)}$ be $dB^{(H)}$-integrable and define

$$X(t) = \int_0^t \psi(s)dB^{(H)}(s)$$

and

$$Y(t) = X^2(t).$$

By (3.25) and (3.26) we have

$$M_{t+}D_{t+}X(t) = \int_0^t M_tD_t\psi(s)dB^{(H)}(s) + M^2(\psi\chi_{[0,t]})_t, \tag{4.2}$$

where $M^2(\psi\chi_{[0,t]})_t = M^2(\psi\chi_{[0,t]})(t)$. Then by Theorem 3.11 and by Proposition 2.12 we have

$$dY(t) = 2X(t)dX(t) + 2\psi(t)\left(\int_0^t M_tD_t\psi(s)dB^{(H)}(s) + M^2(\psi\chi_{[0,t]})_t\right)dt \tag{4.3}$$

In particular, if $\psi \in L_H^2(\mathbb{R})$, we get

$$dY(t) = 2X(t)dX(t) + 2\psi(t)M^2(\psi\chi_{[0,t]})_t dt \tag{4.4}$$

By using that $X_1X_2 = \frac{1}{2}[(X_1+X_2)^2 - X_1^2 - X_2^2]$ this gives the following product rule:

Proposition 4.2 (Product rule) *Suppose $\psi_1, \psi_2 \in L^2_H(\mathbb{R})$ and define*

$$X_i(t) = \int_0^t \psi_i(s) dB^{(H)}(s); \quad i = 1, 2$$

and

$$Y(t) = X_1(t)X_2(t).$$

Then

$$\begin{aligned} dY(t) &= X_1(t)dX_2(t) + X_2(t)dX_1(t) \\ &\quad + \left\{\psi_1(t)M^2(\psi_2\chi_{[0,t]})_t + \psi_2(t)M^2(\psi_1\chi_{[0,t]})_t\right\} dt \end{aligned} \tag{4.5}$$

Corollary 4.3 (Integration by parts) *Let $X_i(t)$, $i = 1, 2$, be as in Proposition 4.2. Then*

$$\begin{aligned} \int_0^t X_1(s)dX_2(s) &= X_1(t)X_2(t) - \int_0^t X_2(s)dX_1(s) \\ &\quad - \int_0^t \left\{\psi_1(s)M^2(\psi_2\chi_{[0,s]})_s + \psi_2(s)M^2(\psi_1\chi_{[0,s]})_s\right\} ds. \end{aligned} \tag{4.6}$$

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